ISRAEL JOURNAL OF MATHEMATICS 92 (1995), 285–293

MONOTONE ITERATIVE METHOD FOR SEMILINEAR ELLIPTIC SYSTEMS OF OPERATOR-DIFFERENTIAL EQUATIONS

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ABSTRACT

Existence of extremal solutions of semilinear elliptic systems of operator-differential equations is proved. The extremal solutions are obtained via monotone iterates.

1. Introduction

The object of this paper is to study the existence of minimal and maximal solutions of boundary value problems (BVP) for semilinear elliptic systems of operator-differential equations. These extremal solutions are obtained as limits of monotone sequences. A linear modification of the problem under consideration is given. The unique solutions of these modified problems are elements of sequences which are convergent in appropriate spaces.

* Supported by the Bulgarian Ministry of Science and Education under Grant MM–7/1991.

** Supported by the International Science Foundation.

Received March 22, 1994

We note that this method of linearization for elliptic equations was introduced in [2]. For related results in other areas we refer to the monograph [3].

2. Preliminary notes

Let $\Omega \subset R^n$, $n \geq 1$ be a bounded domain with a boundary $\partial\Omega$ and $\overline{\Omega} = \Omega \cup \partial\Omega$.

We consider a boundary value problem for a semilinear elliptic system of operator-differential equations

$$Lu = fu \quad \text{in } \Omega, \tag{1}$$

$$u|_{\partial\Omega} = 0, \tag{2}$$

where $u = (u_1, ..., u_m)$: $\overline{\Omega} \to R^m$, $m \geq 1$; f: $C(\overline{\Omega}, R^m) \to C(\overline{\Omega}, R^m)$. The operator L is defined by $Lu = (L_1u_1, ..., L_mu_m)$: $\Omega \to R^m$, where:

$$(L_ku_k)(x) = \sum_{i,j=1}^{n} a_{ij}^k(x) \frac{\partial^2 u_k(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i^k(x) \frac{\partial u_k(x)}{\partial x_i},$$

$$x \in \Omega; \quad k = 1, ..., m; \quad a_{ij}^k = a_{ji}^k \quad \text{for } i, j = 1, ..., n.$$

For two vectors $a, b \in R^m$ we write $a \leq b$ ($a \geq b$) if $a_k \leq b_k$ ($a_k \geq b_k$) for each k, $k = 1, ..., m$. For two functions φ, ψ: $D \to R^m$ we write $\varphi \leq \psi$ ($\varphi \geq \psi$) on D, if $\varphi(x) \leq \psi(x)$ ($\varphi(x) \geq \psi(x)$) for each point $x \in D$.

For the symmetric matrix A we write $A \geq 0$ ($A > 0$), if A is positive semidefinite (definite).

In the sequel we need the following simple lemma.

Lemma 1: *Let* $(a_{ij})_{i,j=1,\dots,n} \geq 0$, $(b_{ij})_{i,j=1,\dots,n} \geq 0$ *be symmetric matrices. Then,*

$$\sum_{i,j=1}^{n} a_{ij} b_{ij} \geq 0.$$

Now we state the classical theorem of Schauder.

Theorem 1 ([5]): *Let the following conditions hold:*

1. All the coefficients and the right-hand side of the equation

$$\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x) \frac{\partial u(x)}{\partial x_i} + a(x)u(x) = f(x), \ x \in \Omega, \tag{3}$$

are of the class $C^\alpha(\overline{\Omega}, R)$, $\alpha \in (0,1)$.

2. The matrix $(a_{ij}(x))_{i,j=1,\dots,n}$ *is symmetric and positive definite for* $x \in \overline{\Omega}$.

3. $a(x) < 0$, $x \in \overline{\Omega}$.

4. $\partial\Omega \in C^{2,\alpha}$.

Then, equation (3) subject to the boundary condition $u|_{\partial\Omega} = 0$ *possesses a unique solution* u *and* $u \in C^{2,\alpha}(\overline{\Omega}, R)$.

We introduce the *following assumptions*:

H1. There exist functions $v^0, w^0 \in C(\overline{\Omega}, R^m) \cap C^2(\Omega, R^m)$ such that

$$Lv^0 \ \geq fv^0 \quad \text{in } \Omega, \quad v^0|_{\partial\Omega} \ \leq \ 0,$$

$$Lw^0 \ \leq fw^0 \quad \text{in } \Omega, \quad w^0|_{\partial\Omega} \ \geq \ 0,$$

and $v^0 \ \leq \ w^0$ on $\overline{\Omega}$.

H2. $f\colon \{u \in C(\overline{\Omega}, R^m)\colon v^0 \leq u \leq w^0 \text{ on } \overline{\Omega}\} \to C(\overline{\Omega}, R^m)$.

H3. $\left(a_{ij}^k(x)\right)_{i,j=1,\dots,n} \geq 0$, $x \in \Omega$, $k = 1, \dots, m$.

3. Main results

LEMMA 2: *Let the following conditions hold:*

1. Assumptions H1 – H3 *are satisfied.*

2. $v, w \in C(\overline{\Omega}, R^m) \cap C^2(\Omega, R^m)$ *and*

$$(Lv)(x) = (fv^0)(x) + g(x, v(x)), \quad x \in \Omega, \tag{4}$$

$$v(x) = 0, \quad x \in \partial\Omega, \tag{5}$$

$$(Lw)(x) = (fw^0)(x) + h(x, w(x)), \quad x \in \Omega,$$

$$w(x) = 0, \quad x \in \partial\Omega.$$

3. $g\colon (\overline{\Omega} \times R^m) \to R^m$, $g(x, v(x)) = (g_1(x, v_1(x)), \dots, g_m(x, v_m(x)))$ *is such that if* $v_k(\tilde{x}) < v_k^0(\tilde{x})$ *for some* $\tilde{x} \in \Omega$, *then* $g_k(\tilde{x}, v_k(\tilde{x})) < 0$.

4. $h\colon (\overline{\Omega} \times R^m) \to R^m$, $h(x, w(x)) = (h_1(x, w_1(x)), \dots, h_m(x, w_m(x)))$ *is such that if* $w_k(\tilde{x}) > w_k^0(\tilde{x})$ *for some* $\tilde{x} \in \Omega$, *then* $h_k(\tilde{x}, w_k(\tilde{x})) > 0$.

Then we have

$$v \ \geq \ v^0, \quad w \ \leq \ w^0 \quad \text{on } \overline{\Omega}.$$

Proof: We prove that $v \geq v^0$ on $\overline{\Omega}$. Suppose that this conclusion is not true, that is, $v_k(x) < v_k^0(x)$ for some $x \in \overline{\Omega}$ and $k \in \{1, ..., m\}$.

Let $y \in \overline{\Omega}$ be such that

$$v_k(y) - v_k^0(y) = \min_{x \in \overline{\Omega}} (v_k(x) - v_k^0(x)).$$

From (5) and the definition of v^0 it follows that $y \in \Omega$ and therefore

$$\left. \frac{\partial (v_k - v_k^0)}{\partial x_i} \right|_{x=y} = 0, \quad i = 1, ..., n,$$

$$\left(\left. \frac{\partial^2 (v_k - v_k^0)}{\partial x_i \partial x_j} \right|_{x=y} \right)_{i,j=1,\dots,n} \geq 0 \ .$$

From Lemma 1 it follows that

$$(L_k v_k)(y) \geq (L_k v_k^0)(y). \tag{6}$$

From (4), (6) and the definition of v^0 it follows that $g_k(y, v_k(y)) \geq 0$, which is a contradiction since $v_k(y) < v_k^0(y)$. Therefore, $v \geq v^0$ on $\overline{\Omega}$.

Analogously, we can prove that $w \leq w^0$ on $\overline{\Omega}$. ∎

LEMMA 3: *Let the following conditions hold:*

1. Assumptions H2, H3 *are satisfied.*

2. There exist functions $\xi^l \in C(\overline{\Omega}, R^m)$, $\eta^l \in C(\overline{\Omega}, R^m) \cap C^2(\Omega, R^m)$, $l = 1, 2$ *such that* $v^0 \leq \xi^1 \leq \xi^2 \leq w^0$ *on* $\overline{\Omega}$,

$$(L\eta^l)(x) = (f\xi^l)(x) + q(x, \xi^l(x), \eta^l(x)), \quad x \in \Omega, \quad l = 1, 2,$$

$$\eta^l(x) = 0, \quad x \in \partial\Omega, \quad l = 1, 2.$$

3. q: $(\overline{\Omega} \times R^m \times R^m) \to R^m$,

$$q(x, \xi(x), \eta(x)) = (q_1(x, \xi_1(x), \eta_1(x)), ..., q_m(x, \xi_m(x), \eta_m(x)))$$

is such that if

$$\beta^l, \gamma^l \in C(\overline{\Omega}, R^m), \quad v^0 \leq \beta^1 \leq \beta^2 \leq w^0 \quad \text{on } \overline{\Omega},$$

$$\beta^1|_{\partial\Omega} \leq 0 \leq \beta^2|_{\partial\Omega}, \quad \gamma^1|_{\partial\Omega} = \gamma^2|_{\partial\Omega} = 0, \quad \gamma_k^1(\tilde{x}) > \gamma_k^2(\tilde{x})$$

for some $\tilde{x} \in \Omega$, *then*

$$(f\beta^2)_k(\tilde{x}) - (f\beta^1)_k(\tilde{x}) < q_k(\tilde{x}, \beta_k^1(\tilde{x}), \gamma_k^1(\tilde{x})) - q_k(\tilde{x}, \beta_k^2(\tilde{x}), \gamma_k^2(\tilde{x})).$$

Then, we have that

$$\eta^1 \le \eta^2 \quad on\ \overline{\Omega}.$$

Proof of Lemma 3: The proof is analogous to that of Lemma 2. We omit it here.

We introduce the *following assumptions*:

H4. There exist functions $v^0, w^0 \in C^\alpha(\overline{\Omega}, R^m) \cap C^2(\Omega, R^m)$, $\alpha \in (0,1)$, such that

$$Lv^0 \ge fv^0 \quad \text{in } \Omega, \quad v^0|_{\partial\Omega} \le 0,$$

$$Lw^0 \le fw^0 \quad \text{in } \Omega, \quad w^0|_{\partial\Omega} \ge 0$$

and $v^0 \le w^0$ on $\overline{\Omega}$.

H5. If $u \in C^\alpha(\overline{\Omega}, R^m)$, then $fu \in C^\alpha(\overline{\Omega}, R^m), \alpha \in (0,1)$.

THEOREM 2: *Let the following conditions hold:*

1. *Assumptions* H4, H5 *are satisfied.*

2. $a_{ij}^k(x)$, $x \in \overline{\Omega}$ *satisfy the Lipschitz condition,* $i, j = 1, ..., n$; $k = 1, ..., m$; $a_i^k \in C^\alpha(\overline{\Omega}, R^m)$, $i = 1, ..., n$; $k = 1, ..., m$; $\alpha \in (0,1)$ *and the matrices* $(a_{ij}^k(x))_{i,j=1,\dots,n}$ *are positive definite for* $x \in \overline{\Omega}$, $k = 1, ..., m$.

3. *The operator* f *maps* $\{u \in L_2(\overline{\Omega}, R^m)$: $v^0 \le u \le w^0$ *on* $\overline{\Omega}\}$ *into* $L_2(\overline{\Omega}, R^m)$ *uniformly, continuously and the restriction of* f *maps* $W_2^2(\Omega, R^m)$ *into* $W_2^2(\Omega, R^m)$ *continuously.*

4. *For each* $\beta^1, \beta^2 \in C^\alpha(\overline{\Omega}, R^m)$ *such that* $v^0 \le \beta^1 \le \beta^2 \le w^0$ *on* $\overline{\Omega}$, *the following inequality is valid:*

$$f\beta^2 - f\beta^1 \le M(\beta^2 - \beta^1), \tag{7}$$

where $M > 0$ *is a constant.*

5. *The constant* M *does not belong to the spectrum of the operator* L *with the boundary condition (2).*

6. $\partial\Omega \in C^{2,\alpha}$.

Then, there exist monotone sequences $\{v^s\}, \{w^s\}$ *which converge point-wise and in* $W_2^2(\Omega, R^m)$ *to* v^∞ *and* w^∞ *respectively. Moreover,* $v^\infty, w^\infty \in C^{2,\alpha}(\overline{\Omega}, R^m)$ *and they are minimal and maximal solutions of the BVP (1), (2) in* *the following sense: if* $u \in C(\overline{\Omega}, R^m) \cap C^2(\Omega, R^m)$ *is a solution of the BVP (1), (2) and* $v^0 \le u \le w^0$ *on* $\overline{\Omega}$, *then*

$$v^\infty \le u \le w^\infty \quad on\ \overline{\Omega}.$$

Proof: We define the sequences $\{v^s\}$ and $\{w^s\}$ as follows:

$$(8)\qquad \begin{cases} Lv^{s+1} - Mv^{s+1} = fv^s - Mv^s & \text{in } \Omega, \\ v^{s+1}|_{\partial\Omega} = 0 \end{cases}$$

and

$$(9)\qquad \begin{cases} Lw^{s+1} - Mw^{s+1} = fw^s - Mw^s & \text{in } \Omega, \\ w^{s+1}|_{\partial\Omega} = 0, \end{cases}$$

$s = 0, 1, \dots$.

By Theorem 1 it follows that, for $s = 0$, BVP's (8) and (9) possess unique solutions v^1 and w^1 respectively, and $v^1, w^1 \in C^{2,\alpha}(\overline{\Omega}, R^m)$. Moreover, by Lemmas 2 and 3 it follows that

$$v^0 \le v^1 \le w^1 \le w^0 \quad \text{on } \overline{\Omega}.$$

Analogously, for $s = 1$ we get $v^2, w^2 \in C^{2,\alpha}(\overline{\Omega}, R^m)$ and

$$v^1 \le v^2 \le w^2 \le w^1 \quad \text{on } \overline{\Omega}.$$

Thus we obtain two sequences

$$v^0 \le v^1 \le v^2 \le \cdots \quad \text{on } \overline{\Omega}$$

and

$$w^0 \ge w^1 \ge w^2 \ge \cdots \quad \text{on } \overline{\Omega}$$

such that $v^s \le w^r$ on $\overline{\Omega}$ for each $s, r = 0, 1, 2, \dots$.

Let $L^M = L - M$. For each $s_1, s_2 \ge 1$, (8) gives us

$$(10)\qquad L^M(v^{s_1} - v^{s_2}) = (fv^{s_1-1} - fv^{s_2-1}) - M(v^{s_1-1} - v^{s_2-1}) \text{ in } \Omega.$$

We have that

$$||v^{s_1} - v^{s_2}||_{L_2(\Omega, R^m)} \to 0 \quad \text{as } s_1, s_2 \to \infty$$

and

$$||(fv^{s_1-1} - fv^{s_2-1}) - M(v^{s_1-1} - v^{s_2-1})||_{L_2(\Omega,R^m)} \to 0 \quad \text{as } s_1, s_2 \to \infty.$$

Therefore, by the second basic inequality in the theory of the linear elliptic equations (Lemma 8.1, Chapter 3, [4]) it follows that

$$||v^{s_1} - v^{s_2}||_{W_2^2(\Omega,R^m)} \to 0 \quad \text{as } s_1, s_2 \to \infty.$$

Thus, the sequence $\{v^s\}$ is convergent in $W_2^2(\Omega, R^m)$. Passing to limit as $s \to \infty$ in (8), we get that v^∞ is a weak solution of the BVP

$$L^M v^\infty = fv^\infty - Mv^\infty \quad \text{in } \Omega, \tag{11}$$

$$v^\infty|_{\partial\Omega} = 0, \tag{12}$$

in the class $W_2^2(\Omega, R^m)$. Hence, it follows that $v^\infty \in C^{1,\varepsilon}(\overline{\Omega}, R^m)$ for some $\varepsilon > 0$ (see Theorem 15.1, Chapter 3, [4]). Therefore, the right-hand side of (11) is of the class $C^\alpha(\overline{\Omega}, R^m)$. The BVP (11), (12) has a unique solution by condition 5 of our theorem. Consequently, $v^\infty \in C^{2,\alpha}(\overline{\Omega}, R^m)$ is a classical solution of the BVP (1), (2) by Theorem 1.

Analogously, we prove that $w^\infty \in C^{2,\alpha}(\overline{\Omega}, R^m)$ is a classical solution of the BVP (1), (2).

Now we prove that if $u \in C(\overline{\Omega}, R^m) \bigcap C^2(\Omega, R^m)$ is a solution of the BVP (1), (2) such that $v^0 \le u \le w^0$ on $\overline{\Omega}$, then $v^\infty \le u \le w^\infty$ on $\overline{\Omega}$. If we suppose that $v^s \le u \le w^s$ on $\overline{\Omega}$ for some s , $s = 0, 1, \ldots$, then it follows that $v^{s+1} \le u \le w^{s+1}$ on $\overline{\Omega}$. Therefore, our conclusion holds by inductive arguments. ∎

Remark 1: From the proof of Theorem 2 and from Theorem 15.1, Chapter 3 of [4] it follows that if the conditions of Theorem 2 hold, then each weak solution of the BVP (1), (2) in the class $W_2^1(\Omega, R^m)$ belongs to the class $C^{2,\alpha}(\overline{\Omega}, R^m)$ and it is a classical solution of the BVP (1), (2). ∎

Remark 2: From Chapter 3, §5 of [4], it follows that condition 5 of Theorem 2 is valid if

$$a_i^k(x) = \sum_{j=1}^{n} \frac{\partial a_{ij}^k(x)}{\partial x_j}, \quad i = 1, \ldots, n; \quad k = 1, \ldots, m; \quad x \in \Omega,$$

or M is sufficiently large, or $\operatorname{mes}\Omega$ is sufficiently small. The corresponding estimates can be obtained from [4]. ∎

Remark 3: If the operator f is reduced to a function, that is, $(fu)(x) = F(x, u(x))$, then the assumption H5 and condition 3 of Theorem 2 are reduced to the following: the function

$$F\colon (D = \{(x,z)\colon x \in \overline{\Omega},\quad v^0(x) \le z \le w^0(x),\ x \in \overline{\Omega}\}) \to R^m$$

has derivatives of first order which satisfy the Lipschitz condition on D. ∎

Example: We consider the BVP for the nonlinear stationary Schrödinger equation

$$\Delta\psi = \varphi\,(|\psi|^2)\,\psi \quad \text{in} \quad \Omega, \tag{13}$$

$$\psi|_{\partial\Omega} = 0, \tag{14}$$

where $\psi\colon \overline{\Omega} \to \mathbb{C}$, $\varphi\colon [0, +\infty) \to (-\infty, 0]$, the function φ is nonincreasing and its derivative satisfies the local Lipschitz condition. Separating the real and imaginary parts of the BVP (13), (14), we obtain the two-dimensional system

$$\Delta u = \varphi\,(|u|^2)\,u \quad \text{in}\ \Omega,$$

$$u|_{\partial\Omega} = 0.$$

Let $v^0 = 0$ on $\overline{\Omega}$ and let $w^0 \in C^\alpha(\overline{\Omega}, R^2) \bigcap C^2(\Omega, R^2)$ satisfy the inequalities

$$\Delta w^0 \ \le\ \varphi\,(|w^0|^2)\,w^0 \quad \text{in}\ \Omega,$$

$$w^0|_{\partial\Omega} \ge 0.$$

Then all the conditions of Theorem 2 are fulfilled, where M is an arbitrary positive number. Thus, we obtain a monotone sequence which converges to the solution of the BVP (13), (14). ∎

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